VERTICAL LOADING ON A STRAIGHT BOUNDARY OF AN ORTHOTROPIC PLATE

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CHAPTER I

INTRODUCTION

Many of the materials in common usage today are orthotropic in their elastic properties. The different types of wood are the most familiar; another less familiar is rolled aluminum in which the rolling process seems to alter the elastic properties of the aluminum and produce one modulus of elasticity in the direction of the rolling and another perpendicular to the direction of rolling. As an illustration of the properties of an orthotropic material consider Sitka spruce: the modulus of elasticity parallel to the grain is 1.679×10^6 lbs. per square inch while the modulus of elasticity perpendicular to the grain is 0.076×10^6 lbs. per square inch. It is evident that serious discrepancies could arise if the orthogonality of the elastic properties of a material such as Sitka spruce were not taken into account when it is put to some practical use. This is especially true when a material is to be used where it will be stressed close to its elastic limit such as in an aircraft. Due to the need to eliminate all excess weight, a member is designed so that the material is used to its utmost advantage. It might be added that quite often for some materials the limiting stress is not determined by the modulus of elasticity but by the modulus of rigidity, that is, the shear stress reaches its limiting value before the normal stresses reach their limiting values.

Consider then a semi-infinite plate oriented so that one of the axes of elastic symmetry is parallel to the boundary of the plate. Such a plate might be a large piece of plywood. Let the axis along the

boundary of the plate be the x-axis and let the y-axis extend in the positive direction into the plate. The plate will be subjected to a load which will act in the plane of the plate and perpendicular to the boundary of the plate. The resulting condition will be treated as plane stress.

The components of stress and strain in an orthotropic plate in a state of plane stress are connected by the following relations:[1]

$$e_{xx} = \frac{\partial u}{\partial x} = \frac{1}{E_x} X_x - \frac{\sigma_{yx}}{E_y} Y_y, \tag{1}$$

$$e_{yy} = \frac{\partial v}{\partial y} = \frac{-\sigma_{xy}}{E_x} + \frac{1}{E_y} Y_y,$$

$$e_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{1}{\mu_{xy}} X_y$$

where the x- and y-axes are the axes of elastic symmetry. In these equations E_x and E_y are Young's moduli of elasticity in the x- and y-directions respectively. Poisson's ratio σ_{xy} is the ratio of the contraction parallel to the y-axis to the extension parallel to the x-axis associated with a tension parallel to the x-axis and similarly for σ_{yx}. The quantity μ_{xy} is the modulus of rigidity associated with the directions of x and y. It should be mentioned that these

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[1] H. W. March, Stress-Strain Relations in Wood and Plywood Considered as Orthotropic Materials, Forest Products Laboratory Report No. R1503, February, 1944, p. 2.

relations hold only for small deformations where all of the stresses produced are within the elastic limits of the material.

The equations of equilibrium for plane stress are

$$\frac{\partial}{\partial x} X_x + \frac{\partial}{\partial y} X_y = 0,$$
$$\frac{\partial}{\partial x} X_y + \frac{\partial}{\partial y} Y_y = 0. \tag{2}$$

These equations are deduced from the conditions of static equilibrium with no reference to the relations between the stresses and strains. They will be satisfied by a stress function[2] F(x,y) such that

$$X_x = \frac{\partial^2 F}{\partial y^2}, \quad Y_y = \frac{\partial^2 F}{\partial x^2}, \quad X_y = \frac{-\partial^2 F}{\partial x \partial y}. \tag{3}$$

The displacements u(x,y) and v(x,y) in the x- and y-directions respectively are necessarily continuous and single-valued. From this condition the compatibility equation for plane stress,

$$\frac{\partial^2 e_{xx}}{\partial y^2} + \frac{\partial^2 e_{yy}}{\partial x^2} = \frac{\partial^2 e_{xy}}{\partial x \partial y} \tag{4}$$

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[2] G. B. Airy, Report of the British Association for the Advancement of Science, 1862.

is derived.

By substituting equations (3) into equations (1) and using equation (4) with the relation[3]

$$\frac{\sigma_{xy}}{E_x} = \frac{\sigma_{yx}}{E_y} \tag{5}$$

the differential equation

$$\frac{\partial^4 F}{\partial \eta^4} + 2K \frac{\partial^4 F}{\partial \eta^2 \partial x^2} + \frac{\partial^4 F}{\partial x^4} = 0 \tag{6}$$

is obtained in which[4]

$$K = \frac{(E_x E_y)^{1/2}}{2}\left(\frac{1}{\mu_{xy}} - \frac{2\sigma_{xy}}{E_x}\right) \tag{7}$$

and

$$\eta = \varepsilon y \tag{8}$$

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[3]March, op, cit., p. 7.

[4]For wood or plywood K, as defined here, is probably always greater than 1.

with

$$\varepsilon = \left(\frac{E_x}{E_y}\right)^{1/4}. \tag{9}$$

The solution of (6) is facilitated by taking F(x,y) to be of the form

$$F = R\left[F_1(x + i\alpha\varepsilon y) + F_2(x + i\beta\varepsilon y)\right] \tag{10}$$

where

$$\alpha = \sqrt{K + \sqrt{K^2 - 1}}\,, \quad \beta = \sqrt{K - \sqrt{K^2 - 1}} \tag{11}$$

i is the imaginary unit, and the letter R means that the real part of the expression in the bracket is to be taken. The functions F_1 and F_2 are any analytic functions of the complex variables $x + i\alpha\varepsilon y$ and $x + i\beta\varepsilon y$ respectively.

CHAPTER II

GENERAL EQUATIONS FOR STRESSES AND DISPLACEMENTS

Two types of problems are to be considered. They are:

Type I: Problems in which only the stresses are assigned on the boundary, and

Type II: Problems in which certain stresses are assigned on the boundary together with certain displacements on the boundary.

In both types the load will be symmetric with respect to the y-axis but shall not be periodic. Also the shear stress shall vanish on the entire boundary in both types.

Assume then a stress function in the form

$$F = R\left\{\sum_{m=1}^{\infty}\left(A_m e^{im z_1} + B_m e^{im z_2}\right)\right\} \tag{12}$$

where

$$A_m = A_{m1} + i A_{m2}, \quad B_m = B_{m1} + i B_{m2},$$

$$z_1 = x + i\alpha\varepsilon y, \quad z_2 = x + i\beta\varepsilon y.$$

Differentiation with respect to the proper variables yields

$$X_x = \frac{\partial^2 F}{\partial y^2} = R\left\{\sum_{m=1}^{\infty}\left(\alpha^2\varepsilon^2 m^2 A_m e^{im z_1} + \beta^2\varepsilon^2 m^2 B_m e^{im z_2}\right)\right\}, \tag{13}$$

$$Y_y = \frac{\partial^2 F}{\partial x^2} = R\left\{\sum_{m=1}^{\infty}\left(-m^2 A_m e^{im z_1} - m^2 B_m e^{im z_2}\right)\right\},$$

$$X_y = \frac{-\partial^2 F}{\partial x \partial y} = R\left\{\sum_{m=1}^{\infty}\left(i\alpha\varepsilon m^2 A_m e^{im z_1} + i\beta\varepsilon m^2 B_m e^{im z_2}\right)\right\}.$$

Noting that

$$e^{im z_1} = e^{-m\alpha\varepsilon y}(\cos mx + i \sin mx),$$

$$e^{im z_2} = e^{-m\beta\varepsilon y}(\cos mx + i \sin mx),$$

and taking the real part of the expression in the parentheses makes

$$\begin{aligned}
X_x &= \sum_{m=1}^{\infty}\Big[m^2\alpha^2\varepsilon^2 e^{-m\alpha\varepsilon y}(A_{m1}\cos mx - A_{m2}\sin mx) \\
&\qquad + m^2\beta^2\varepsilon^2 e^{-m\beta\varepsilon y}(B_{m1}\cos mx - B_{m2}\sin mx)\Big], \\
Y_y &= \sum_{m=1}^{\infty}\Big[-m^2 e^{-m\alpha\varepsilon y}(A_{m1}\cos mx - A_{m2}\sin mx) \\
&\qquad - m^2 e^{-m\beta\varepsilon y}(B_{m1}\cos mx - B_{m2}\sin mx)\Big], \\
X_y &= \sum_{m=1}^{\infty}\Big[m^2\alpha\varepsilon e^{-m\alpha\varepsilon y}(A_{m2}\cos mx + A_{m1}\sin mx) \\
&\qquad + m^2\beta\varepsilon e^{-m\beta\varepsilon y}(B_{m2}\cos mx + B_{m1}\sin mx)\Big].
\end{aligned} \tag{14}$$

The stress Y_y is symmetric about the y-axis and in order to satisfy this condition it is necessary that

$$A_{m2} = B_{m2} = 0.$$

The stress function then reduces to

$$F = R\left\{\sum_{m=1}^{\infty}\left(A_{m1}e^{im z_1} + B_{m2}e^{im z_2}\right)\right\}$$

or

$$F = \sum_{m=1}^{\infty}\left[\left(A_{m1}e^{-m\alpha\varepsilon y} + B_{m1}e^{-m\beta\varepsilon y}\right)\cos m x\right]$$

The stress function is periodic now in x and in the form of a Fourier series. The stresses are to be non-periodic in x so transform the stress function into a Fourier integral, that is,

$$F = \frac{2}{\pi}\int_0^{\infty} dm \int_0^{\infty}\left[A(t)e^{-m\alpha\varepsilon y} + B(t)e^{-m\beta\varepsilon y}\right]\cos m x \cos m t \, dt$$

where A(t) and B(t) are to be determined from the boundary conditions. Differentiation yields

$$X_x = \frac{2}{\pi}\int_0^{\infty} dm \int_0^{\infty} m^2\left[A(t)\alpha^2\varepsilon^2 e^{-m\alpha\varepsilon y} + B(t)\beta^2\varepsilon^2 e^{-m\beta\varepsilon y}\right]\cos m x \cos m t \, dt,$$

$$Y_y = -\frac{2}{\pi}\int_0^\infty dm \int_0^\infty m^2\left[A(t)e^{-m\alpha\varepsilon y} + B(t)e^{-m\beta\varepsilon y}\right]\cos mx \cos mt\, dt,$$

$$X_y = -\frac{2}{\pi}\int_0^\infty dm \int_0^\infty m^2\left[A(t)\alpha\varepsilon e^{-m\alpha\varepsilon y} + B(t)\beta\varepsilon e^{-m\beta\varepsilon y}\right]\cos mx \cos mt\, dt. \tag{16}$$

The stipulation that the shear vanish on the boundary makes it necessary that

$$A(t)\alpha + B(t)\beta = 0$$

or

$$B(t) = -\frac{\alpha}{\beta}A(t) \tag{17}$$

Equations (16) reduce then to

$$X_x = \frac{2}{\pi}\int_0^\infty dm \int_0^\infty m^2 A(t)\left[\alpha^2\varepsilon^2 e^{-m\alpha\varepsilon y} - \alpha\beta\varepsilon^2 e^{-m\beta\varepsilon y}\right]\cos mx \cos mt\, dt,$$

$$Y_y = -\frac{2}{\pi}\int_0^\infty dm \int_0^\infty m^2 A(t)\Big[e^{-m\alpha\varepsilon y} - \frac{\alpha}{\beta} e^{-m\beta\varepsilon y}\Big] \cos mx \cos mt \, dt,$$

$$X_y = -\frac{2}{\pi}\int_0^\infty dm \int_0^\infty m^2 A(t)\Big[\alpha\varepsilon e^{-m\alpha\varepsilon y} - \alpha\varepsilon e^{-m\beta\varepsilon y}\Big] \cos mx \cos mt \, dt, \tag{18}$$

which are the general equations for the stresses in the form of a Fourier integral.

From equations (1)

$$\frac{\partial u}{\partial x} = \frac{2}{\pi}\int_0^\infty dm \int_0^\infty m^2 A(t)\Big[(s_{11}\alpha^2\varepsilon^2 - s_{12})e^{-m\alpha\varepsilon y} - (s_{11}\alpha\beta\varepsilon^2 - s_{12})e^{-m\beta\varepsilon y}\Big] \cos mx \cos mt \, dt,$$

$$\frac{\partial v}{\partial x} = \frac{2}{\pi}\int_0^\infty dm \int_0^\infty m^2 A(t)\Big[(s_{12}\alpha^2\varepsilon^2 - s_{22})e^{-m\alpha\varepsilon y} - (s_{12}\alpha\beta\varepsilon^2 - s_{22}\frac{\alpha}{\beta})e^{-m\beta\varepsilon y}\Big] \cos mx \cos mt \, dt,$$

where

$$s_{11} = \frac{1}{E_x}, \quad s_{12} = \frac{-\sigma_{xy}}{E_x} = \frac{-\sigma_{yx}}{E_y}, \quad s_{22} = \frac{1}{E_y}.$$

Integrating with respect to x or y as the case may be yields

$$u = \frac{2}{\pi}\int_0^\infty dm \int_0^\infty m A(t)\Big[(S_{11}\alpha^2\varepsilon^2 - S_{12})e^{-m\alpha\varepsilon y}$$
$$- (S_{11}\alpha\beta\varepsilon^2 - S_{12})e^{-m\beta\varepsilon y}\Big]\cos mx \cos mt\, dt + u_0(y),$$

$$v = -\frac{2}{\pi}\int_0^\infty dm \int_0^\infty m A(t)\Big[\Big(S_{12}\alpha\varepsilon - \frac{S_{22}}{\alpha\varepsilon}\Big)e^{-m\alpha\varepsilon y} \tag{19}$$
$$- \Big(S_{12}\beta\varepsilon - \frac{S_{22}}{\beta\varepsilon}\Big)e^{-m\beta\varepsilon y}\Big]\cos mx \cos mt\, dt + v_0(x),$$

where the functions $u_0(y)$ and $v_0(x)$ are determined from

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{1}{\mu_{xy}} X_y$$

in equations (1). These are the general equations for the displacements $u(x,y)$ and $v(x,y)$ in the form of a Fourier integral.

CHAPTER III

SOLUTIONS FOR PROBLEMS OF TYPE I

Consider now problems of Type I, that is, problems in which only the stresses are assigned on the boundary. In all of these problems the shear stress will vanish on the boundary making

$$X_y = 0 \quad when \quad y = 0 \tag{20}$$

which will permit the use of the equations obtained in the preceding chapter.

For the first problem assign a normal load on the boundary distributed in the manner of

$$-\frac{H}{b^2 + x^2}$$

which is another way of saying that

$$Y_y = \frac{-H}{b^2 + x^2} \tag{21}$$

since the distribution of the stress Y_y along the boundary is equal to the normal load. Then, using equations (18), when y is zero, we have

$$Y_y = \frac{-2}{\pi}\int_0^\infty dm \int_0^\infty m^2 A(t)\left[1 - \frac{\alpha}{\beta}\right] \cos mx \cos mt \, dt$$

From this equation it follows that

$$m^2\left(1-\frac{\alpha}{\beta}\right)A(t) = \frac{H}{b^2+t^2}$$

or

$$A(t) = \frac{H\beta}{m^2(\beta-\alpha)(b^2+t^2)}.$$

Substituting this value for A(t) in equations (18) makes

$$X_x = \frac{2H}{\pi(\beta-\alpha)}\int_0^\infty dm\int_0^\infty\Big[\alpha^2\beta\varepsilon^2 e^{-m\alpha\varepsilon y}$$

$$-\alpha\beta^2\varepsilon^2 e^{-m\beta\varepsilon y}\Big]\frac{\cos mx\cos mt}{b^2+t^2}\,dt,$$

$$Y_y = \frac{-2H}{\pi(\beta-\alpha)}\int_0^\infty dm\int_0^\infty\Big[\beta e^{-m\alpha\varepsilon y}$$

$$-\alpha e^{-m\beta\varepsilon y}\Big]\frac{\cos mx\cos mt}{b^2+t^2}\,dt,$$

$$X_y = \frac{-2H\alpha\varepsilon}{\pi(\beta-\alpha)}\int_0^\infty dm\int_0^\infty\Big[\beta e^{-m\alpha\varepsilon y}$$

$$-\beta e^{-m\beta\varepsilon y}\Big]\frac{\sin mx\cos mt}{b^2+t^2}\,dt.$$

Noting that the integral is in the form of

$$\int_0^\infty \frac{\cos mt}{b^2+t^2}\,dt \doteq \frac{\pi}{2b}e^{-bm} \tag{22}$$

and that $\alpha\beta = 1$, we obtain on integrating with respect to t

$$X_x = \frac{H\varepsilon^2}{b(\beta-\alpha)}\int_0^\infty\left[\alpha e^{-m(b+\alpha\varepsilon y)} - \beta e^{-m(b+\beta\varepsilon y)}\right]\cos mx\,dm,$$

$$Y_y = \frac{-H}{b(\beta-\alpha)}\int_0^\infty\left[\beta e^{-m(b+\alpha\varepsilon y)} - \alpha e^{-m(b+\beta\varepsilon y)}\right]\cos mx\,dm,$$

$$X_y = \frac{-H\varepsilon}{b(\beta-\alpha)}\int_0^\infty\left[e^{-m(b+\alpha\varepsilon y)} - e^{-m(b+\beta\varepsilon y)}\right]\sin mx\,dm.$$

In order to integrate these last integrals notice that

$$\int_0^\infty e^{-km}\cos lm\,dm = \frac{k}{k^2+l^2}$$

and

$$\int_0^\infty e^{-km}\sin lm\,dm = \frac{l}{k^2+l^2}.$$

The use of these yield

$$X_x = \frac{H\varepsilon^2}{b(\beta-\alpha)}\left\{\frac{\alpha(b+\alpha\varepsilon y)}{\left[(b+\alpha\varepsilon y)^2+x^2\right]} - \frac{\beta(b+\beta\varepsilon y)}{\left[(b+\beta\varepsilon y)^2+x^2\right]}\right\},$$

$$Y_y = \frac{-H}{b(\beta-\alpha)}\left\{\frac{\beta(b+\alpha\varepsilon y)}{\left[(b+\alpha\varepsilon y)^2+x^2\right]} - \frac{\alpha(b+\beta\varepsilon y)}{\left[(b+\beta\varepsilon y)^2+x^2\right]}\right\}, \qquad (23)$$

$$X_y = \frac{-H\varepsilon}{b(\beta-\alpha)}\left\{\frac{x}{\left[(b+\alpha\varepsilon y)^2+x^2\right]} - \frac{x}{\left[(b+\beta\varepsilon y)^2+x^2\right]}\right\},$$

which are the stresses produced by a load distributed in the manner of $-H/(b^2+x^2)$.

Now suppose that we have a point load of magnitude P applied at the origin in the positive y direction instead of the one just discussed. It is possible to get the stress equations for the point load P from the stress equations just obtained by allowing the constant b in the load function to approach zero while the total load remains constant. To do this notice that the total load is

$$\int_{-\infty}^{\infty}\frac{H}{b^2+x^2}\,dx = \frac{H}{b}\tan^{-1}\frac{x}{b}\Bigg|_{-\infty}^{\infty} = \frac{H\pi}{b}$$

and let

$$\frac{H\pi}{b} = P.$$

Then

$$H = \frac{Pb}{\pi}.$$

Putting this value in the stress equations (22) and letting b approach zero we have

$$X_x = \frac{P\varepsilon}{\pi(\beta-\alpha)}\left[\frac{\alpha^2\varepsilon^2 y}{(\alpha\varepsilon y)^2+x^2} - \frac{\beta^2\varepsilon^2 y}{(\beta\varepsilon y)^2+x^2}\right],$$

$$Y_y = \frac{-P\varepsilon}{\pi(\beta-\alpha)}\left[\frac{y}{(\alpha\varepsilon y)^2+x^2} - \frac{y}{(\beta\varepsilon y)^2+x^2}\right], \tag{24}$$

$$X_y = \frac{-P\varepsilon}{\pi(\beta-\alpha)}\left[\frac{x}{(\alpha\varepsilon y)^2+x^2} - \frac{x}{(\beta\varepsilon y)^2+x^2}\right].$$

As a check we could investigate these stress equations to see if the shear stress does vanish on the boundary and if

$$\int_{-\infty}^{\infty} Y_y\, dx = -P \quad \text{when } y=d \text{ and } d>0$$

that is, the total force on a boundary $y = d$ of an infinite strip in the plate must equal the force P on the boundary $y = o$ in order for the strip to be in static equilibrium. The shear stress vanishes immediately when $y = o$ and

$$\int_{-\infty}^{\infty} Y_y \, dx = \frac{-P\varepsilon}{\pi(\beta-\alpha)} \int_{-\infty}^{\infty} \left[\frac{y}{(\alpha\varepsilon y)^2 + x^2} - \frac{y}{(\beta\varepsilon y)^2 + x^2} \right] dx$$

$$= \frac{-P\varepsilon}{\pi(\beta-\alpha)} \left[\frac{y}{\alpha\varepsilon y} \tan^{-1}\frac{x}{\alpha\varepsilon y} - \frac{y}{\beta\varepsilon y} \tan^{-1}\frac{x}{\beta\varepsilon y} \right]_{-\infty}^{\infty}$$

$$= -P.$$

Let us also investigate to see if these stresses reduce to the isotropic case. In the isotropic case $\alpha = \beta = \varepsilon = 1$ so noticing that when $\alpha = \beta$ the stresses are indeterminant we hold β constant and using l'Hospital's rule we obtain

$$X_x = \lim_{\alpha \to \beta \to 1} \frac{P\varepsilon}{-\pi} \left\{ \frac{2\alpha\varepsilon^2 x^2 y}{\left[(\alpha\varepsilon y)^2 + x^2\right]^2} \right\}$$

$$= -\frac{2P}{\pi} \frac{\sin^2\theta \cos\theta}{r},$$

$$Y_y = \lim_{\alpha \to \beta \to 1} \frac{P\varepsilon}{\pi} \left\{ \frac{-2\alpha\varepsilon^2 y^3}{\left[(\alpha\varepsilon y)^2 + x^2\right]^2} \right\}$$

$$= -\frac{2P}{\pi} \frac{\cos^3\theta}{r},$$

$$X_y = \lim_{\alpha \to \beta \to 1} \frac{P\varepsilon}{\pi} \left\{ \frac{-2\alpha\varepsilon^2 x y^2}{\left[(\alpha\varepsilon y)^2 + x^2\right]^2} \right\}$$

$$= -\frac{2P}{\pi} \frac{\sin\theta\cos^2\theta}{r},$$

which agrees with the values obtained by Timoshenko if the choice of axes is made to agree[5]. It might be added that the stresses at the origin are infinite as would be expected from a point load at the point of application. Therefore equations (24) are the stress equations in a semi-infinite plate of orthotropic material for the point load P.

The displacement v due to the point load P will be calculated now since it is to be used in the next chapter. The use of equations (1) and (24) yields

$$\frac{\partial v}{\partial y} = \frac{P\varepsilon}{\pi(\beta-\alpha)} \left\{ \left[\frac{(S_{12}\alpha^2\varepsilon^2 - S_{22})y}{(\alpha\varepsilon y)^2 + x^2} \right] - \left[\frac{(S_{12}\beta^2\varepsilon^2 - S_{22})y}{(\beta\varepsilon y)^2 + x^2} \right] \right\}.$$

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[5] S. Timoshenko, Theory of Elasticity, New York, 1934, p. 84.

Integration with respect to y makes

$$v = \frac{P\varepsilon}{2\pi(\beta-\alpha)}\left\{\left(s_{12}-\frac{s_{22}}{\alpha^2\varepsilon^2}\right)\log\left[(\alpha\varepsilon y)^2+x^2\right]\right.$$
$$\left.-\left(s_{12}-\frac{s_{22}}{\beta^2\varepsilon^2}\right)\log\left[(\beta\varepsilon y)^2+x^2\right]\right\}+v_0(x).$$

The function $v_0(x)$ must either be a linear function of x or a constant. It cannot be a linear because v is symmetric with respect to the y-axis. Therefore v_0 is a constant which depends on an assigned deflection at the point (0,d). Then the equation for v due to a point load is

$$v = \frac{P\varepsilon}{2\pi(\beta-\alpha)}\left\{\left(s_{12}-\frac{s_{22}}{\alpha^2\varepsilon^2}\right)\log\left[(\alpha\varepsilon y)^2+x^2\right]\right.$$
$$\left.-\left(s_{12}-\frac{s_{22}}{\beta^2\varepsilon^2}\right)\log\left[(\beta\varepsilon y)^2+x^2\right]\right\}+k \tag{25}$$

where k is the value of v at the point (0,d).

CHAPTER IV

SOLUTIONS FOR PROBLEMS OF TYPE II

Consider now problems of Type II, that is, problems in which certain stresses are assigned on the boundary together with certain displacements on the boundary. Let the assigned stresses in each case be

$$X_y = 0 \quad when \quad y = 0 \tag{26}$$

which will permit the use of the equations developed in the second chapter. Also let the displacements on the boundary in each case be the displacement v. The problem shall be to determine the load necessary to produce each of these boundary conditions.

For the first problem assume a displacement along the boundary

$$v = \frac{H}{b^2 + x^2}. \tag{27}$$

Then from equations (19)

$$v = \frac{-2}{\pi}\int_0^\infty dm \int_0^\infty m A(t)\left[\left(S_{12}\alpha\varepsilon - \frac{S_{22}}{\alpha\varepsilon}\right)e^{-m\alpha\varepsilon y}\right.$$
$$\left. -\left(S_{12}\beta\varepsilon - \frac{S_{22}}{\beta\varepsilon}\right)e^{-m\beta\varepsilon y}\right]\cos mx \cos mt\, dt + v_0(x)$$

and when y = o we have

$$v = \frac{-2}{\pi}\int_0^\infty dm \int_0^\infty m A(t)\Big[(S_{12}\alpha\varepsilon - S_{12}\beta\varepsilon)$$

$$-\left(\frac{S_{22}}{\alpha\varepsilon} - \frac{S_{22}}{\beta\varepsilon}\right)\Big]\cos mt \cos mx \, dt + v_0(x).$$

The function $v_0(x)$ must be zero because it can either be a linear function of x, which is impossible due to the symmetry of v, or a constant, which must be zero so that v will vanish as x becomes infinite. Therefore

$$\frac{H}{b^2+t^2} = A(t)\, m(\alpha\varepsilon - \beta\varepsilon)\left[S_{12} + \frac{S_{22}}{\alpha\beta\varepsilon^2}\right]$$

or

$$A(t) = \frac{H}{m\varepsilon(\alpha-\beta)(b^2+t^2)\left(S_{12} + \frac{S_{22}}{\alpha\beta\varepsilon^2}\right)} \quad . \tag{28}$$

which, by noting that $\alpha\beta = 1$, makes

$$v = \frac{2H\varepsilon^2}{\pi S_{22}(\beta^2-\alpha^2)}\int_0^\infty dm \int_0^\infty \Big[\left(S_{12} - \frac{S_{22}}{\alpha^2\varepsilon^2}\right)e^{-m\alpha\varepsilon y}$$

$$-\left(S_{12} - \frac{S_{22}}{\alpha^2\varepsilon^2}\right)e^{-m\beta\varepsilon y}\Big]\frac{\cos mx \cos mt}{b^2+t^2}\, dt \quad .$$

Using equation (24) we have

$$v = \frac{2H\varepsilon^2}{\pi s_{22}(\beta^2-\alpha^2)}\int_0^\infty\left[\left(\frac{s_{22}}{\alpha^2\varepsilon^2} - s_{12}\right)e^{-m\alpha\varepsilon y}\right.$$

$$\left.-\left(\frac{s_{22}}{\beta^2\varepsilon^2} - s_{12}\right)e^{-m\beta\varepsilon y}\right]\frac{\pi}{2b}e^{-bm}\cos mx\,dx$$

or

$$v = \frac{H\varepsilon^2}{b\,s_{22}(\beta^2-\alpha^2)}\int_0^\infty\left[\left(\frac{s_{22}}{\alpha^2\varepsilon^2} - s_{12}\right)e^{-m(b+\alpha\varepsilon y)}\right.$$

$$\left.-\left(\frac{s_{22}}{\beta^2\varepsilon^2} - s_{12}\right)e^{-m(b+\beta\varepsilon y)}\right]\cos mx\,dx\,.$$

Then remembering that

$$\int_0^\infty e^{-km}\cos lm\,dm = \frac{k}{k^2+l^2}$$

the integration with respect to m yields

$$v = \frac{H\varepsilon^2}{b\,s_{22}(\beta^2-\alpha^2)}\left\{\left[\left(\frac{s_{22}}{\alpha^2\varepsilon^2} - s_{12}\right)\frac{(b+\alpha\varepsilon y)}{(b+\alpha\varepsilon y)^2+x^2}\right]\right.$$

$$\left.-\left[\left(\frac{s_{22}}{\beta^2\varepsilon^2} - s_{12}\right)\frac{(b+\beta\varepsilon y)}{(b+\beta\varepsilon y)^2+x^2}\right]\right\}. \qquad (29)$$

When $y = 0$

$$v = \frac{H\varepsilon^2}{b s_{22}(\beta^2-\alpha^2)}\left[\frac{s_{22}b}{\alpha^2\varepsilon^2(b^2+x^2)} - \frac{s_{22}b}{\beta^2\varepsilon^2(b^2+x^2)}\right]$$

$$= \frac{H}{b^2+x^2}$$

which satisfies the boundary condition equation (27).

In order to find the load distribution consider the stress Y_y where from equations (18) and (28)

$$Y_y = \frac{2H\varepsilon}{\pi s_{22}(\beta^2-\alpha^2)}\int_0^\infty dm\int_0^\infty\left[\beta e^{-m\alpha\varepsilon y}\right.$$

$$\left.-\alpha e^{-m\beta\varepsilon y}\right]\left[\frac{m\cos mx\cos mt}{b^2+t^2}\right]dt.$$

Using equation (22) and integrating with respect to t yields

$$Y_y = \frac{H\varepsilon}{s_{22}b(\beta^2-\alpha^2)}\int_0^\infty\left[\beta e^{-m(b+\alpha\varepsilon y)} - \alpha e^{-m(b+\beta\varepsilon y)}\right]m\cos mx\,dm.$$

In order to integrate this last integral consider

$$\int_0^\infty e^{-km}\cos lm\,dm = \frac{k}{k^2+l^2}.$$

Differentiating each side with respect to k makes

$$\int_0^\infty e^{-km} m \cos lm \, dm = \frac{l^2-k^2}{[l^2+k^2]^2}.$$

We can use this result to obtain

$$Y_y = \frac{H\varepsilon^2}{S_{22}b(\beta^2-\alpha^2)}\left\{\frac{x^2-(b+\alpha\varepsilon y)^2}{\alpha\varepsilon\left[x^2+(b+\alpha\varepsilon y)^2\right]^2}\right.$$

$$\left. - \frac{x^2-(b+\beta\varepsilon y)^2}{\beta\varepsilon\left[x^2+(b+\beta\varepsilon y)^2\right]^2}\right\}.$$

When $y = o$

$$Y_y = \frac{H\varepsilon}{S_{22}b(\beta^2-\alpha^2)}\left\{\frac{x^2-b^2}{\alpha(x^2+b^2)^2} - \frac{x^2-b^2}{\beta(x^2+b^2)^2}\right\}$$

or

$$Y_y = \frac{H\varepsilon(x^2-b^2)}{S_{22}b(\beta+\alpha)(x^2+b^2)^2}.$$

This is the distribution of the stress Y_y along the boundary and therefore the distribution of the load which is a solution to the problem.

Consider now a semi-infinite plate whose boundary is free of shear stress and that under the load the displacement in the y direction is constant, that is,

$$X_y = 0 \quad \text{when } y = 0 \tag{30}$$

$$v = c \quad \text{when } y = 0 \text{ and } -a \leq x \leq a, \tag{31}$$

$$Y_y = 0 \quad \text{when } y = 0 \text{ and } |x| > a. \tag{32}$$

Then the effect of the load is known and the problem is to determine what load will produce these conditions on the boundary of the plate.

In order to determine the load distribution, first suppose that there was more than one point load acting at various points on the boundary of the plate. If the effect of each load taken independently were known then the combined effect of all the loads would be the vector sum of the effect due to each load. Keeping this in mind, consider the total load as a series of point loads which are distributed from a to -a. The point load will vary in magnitude so that the combined effect of all the point loads will produce the given boundary conditions. Each point load will be a function of ξ, its distance from the origin along the x-axis. Then the equation for the displacement v due to a point load a distance ξ from the origin is

$$v = \frac{\varepsilon P(\xi)}{2\pi(\beta - \alpha)} \left\{ \left(s_{12} - \frac{s_{22}}{\alpha^2 \varepsilon^2} \right) \log\left[(\alpha \varepsilon y)^2 + (x - \xi)^2 \right] \right.$$

$$\left. - \left(s_{12} - \frac{s_{22}}{\beta^2 \varepsilon^2} \right) \log\left[(\beta \varepsilon y)^2 + (x - \xi)^2 \right] \right.$$

by using equation (25) where $P(\xi)d\xi$ is the magnitude of the point load acting at the point ξ,0. The constant k in equation (25) will have no effect on the determination of $P(\xi)$ since it gives a constant deflection in the y-direction for every point along the boundary. Hence it needs not be considered here. From this it follows that the equation for the displacement v for a series of loads along the x-axis from a to -a would be

$$v = \int_{-a}^{a} \frac{\varepsilon P(\xi)}{2\pi(\beta-\alpha)} \left\{ \left(S_{12} - \frac{S_{22}}{\alpha^2\varepsilon^2}\right) \log\left[(\alpha\varepsilon y)^2 + (x-\xi)^2\right] - \left(S_{12} - \frac{S_{22}}{\beta^2\varepsilon^2}\right) \log\left[(\beta\varepsilon y)^2 + (x-\xi)^2\right] \right\} d\xi \tag{33}$$

When $y = 0$

$$v = \frac{-S_{22}(\alpha+\beta)}{2\pi\varepsilon} \int_{-a}^{a} P(\xi) \log(x-\xi)\, d\xi$$

the value of which is to be a given constant independent of x between a and -a.[6] Therefore we have an integral equation in which $P(\xi)$ is to be determined.

The problem is similar to one encountered in electrostatic potential in which an infinite conducting strip, whose width is 2a, is perpendicular to the xy-plane.[7] If the strip is electrostatically

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[6]It is easily shown that a uniform loading of the straight boundary from a to -a would not produce a constant deflection under the load, since in this case $P(\xi)$ would be a constant and equation (33) readily integrates to show that v varies with x when y is zero.

[7]O. D. Kellogg, Foundations of Potential Theory, Berlin, 1929, p. 375.

charged then the potential on the strip must be constant, otherwise a flow of charges would be started. It is shown in potential theory that the potential ϕ in the xy-plane due to a long straight uniformily charged wire perpendicular to the plane is logarithmic, that is,

$$\phi = -q \log r$$

where q is the charge per two units of length and r is the radial distance in the xy-plane from the point where the wire intersects the plane. Then think of the strip as a series of wires perpendicular to the xy-plane along the x-axis from a to -a. Let q be the total charge per two units of length of the strip perpendicular to the plane. It is then necessary to determine the distribution of q along x-axis from a to -a so that the potential ϕ will be constant along that part of the x-axis.

If we use the complex variable $z = x + iy$, then the transformation

$$z = \frac{a}{2}\left(\rho + \frac{1}{\rho}\right)$$

maps the charged strip on the x-axis in the z-plane into the unit circle in the ρ-plane. The lower half of the strip maps into the lower half of the circle in the ρ-plane, the upper half of the strip into the upper half of the circle. The rest of the z-plane can be mapped into the region outside of the unit circle in the ρ-plane. Now at great distances from the origin the potential must become negatively infinite like -q log r. At great distances from the origin ρ

becomes infinite like $\frac{2}{a}|z|$, that is like $\frac{2}{a}r$. Hence the function

$$\phi = -q \log \frac{a}{2}|\rho| \tag{34}$$

which is constant on the circle $|\rho| = 1$, will satisfy the requirements for the potential.

It is also shown for two-dimensional potential in potential theory that on the charged strip

$$\frac{\partial \phi}{\partial n_+} - \frac{\partial \phi}{\partial n_-} = -2\pi\sigma \tag{35}$$

where n is the normal and σ is the distribution of the charge q on the surface in the x-direction. On the charged strip the normal is in the direction of y. Therefore equation (35) becomes

$$\frac{\partial \phi}{\partial y_+} - \frac{\partial \phi}{\partial y_-} = -2\pi\sigma \tag{36}$$

where the plus and minus signs mean taking the partial derivative with respect to y from the positive side and negative side respectively. Solving for ρ one obtains

$$\phi = -q \log \frac{1}{2}\left| z \pm \sqrt{z^2 - a^2} \right| . \tag{37}$$

In order to determine whether to use the plus or minus sign before the radical notice that

$$|\rho| = \left| \frac{z}{a} \pm \sqrt{\frac{z^2}{a^2} - 1} \right| \geqq 1$$

so that the values of ρ will not be inside of the unit circle in the ρ-plane, for no points in the z-plane are to be transformed into points inside of the unit circle in ρ-plane. Then when investigating points along the y axis we have for $y = \varepsilon > o$

$$\left| \frac{i\varepsilon}{a} \pm \sqrt{-\frac{\varepsilon^2}{a^2} - 1} \right| = \left| \frac{i\varepsilon}{a} \pm i\sqrt{\frac{\varepsilon^2}{a^2} + 1} \right| \geqq 1$$

which shows that, for this last quantity to be equal to or greater than one, the plus sign must be chosen. For $y = \varepsilon < o$ we have

$$\left| \frac{-i\varepsilon}{a} \pm \sqrt{-\frac{\varepsilon^2}{a^2} - 1} \right| = \left| \frac{-i\varepsilon}{a} \pm i\sqrt{\frac{\varepsilon^2}{a^2} + 1} \right| \geqq 1$$

from which follows the negative sign is to be used when y is negative.

Differentiation yields

$$\frac{\partial \phi}{\partial y_+} = \frac{-q}{2}\left[\frac{i\left(1 + \frac{z}{\sqrt{z^2 - a^2}}\right)}{\left|z + \sqrt{z^2 - a^2}\right|}\right]$$

$$= \frac{-q}{2}\left[\frac{i}{\sqrt{z^2 - a^2}}\right] = \frac{-q}{2}\left[\frac{1}{\sqrt{a^2 - z^2}}\right]$$

which on the strip when y is zero reduces to

$$\frac{\partial \phi}{\partial y_+} = \frac{-q}{2\sqrt{a^2 - x^2}}.$$

Similarly when on the strip

$$\frac{\partial \phi}{\partial y_-} = \frac{-q}{2}\left[\frac{i\left(1 - \frac{z}{\sqrt{z^2 - a^2}}\right)}{z - \sqrt{z^2 - a^2}}\right]$$

$$\frac{\partial \phi}{\partial y_-} = \frac{-q}{2}\left[\frac{-i}{\sqrt{z^2-a^2}}\right] = \frac{q}{2\sqrt{a^2-x^2}} .$$

Then using equation (35) we have

$$\frac{-q}{2\sqrt{a^2-x^2}} - \frac{q}{2\sqrt{a^2-x^2}} = -2\pi\sigma$$

or

$$\sigma = \frac{q}{2\pi\sqrt{a^2-x^2}} .$$

This is the desired distribution of the charge q along the x-axis between a and -a.

Now returning to the original problem and rewriting equation (33) in the form

$$v = \frac{\varepsilon\left(s_{12} - \frac{s_{22}}{\alpha^2\varepsilon^2}\right)}{2\pi(\beta-\alpha)}\int_{-a}^{a} P(\xi)\log\left[(\alpha\varepsilon y)^2 + (x-\xi)^2\right]d\xi$$
$$- \frac{\varepsilon\left(s_{12} - \frac{s_{22}}{\beta^2\varepsilon^2}\right)}{2\pi(\beta-\alpha)}\int_{-a}^{a} P(\xi)\log\left[(\beta\varepsilon y)^2 + (x-\xi)^2\right]d\xi \tag{38}$$

a similarity can be drawn between the two integrals and the potential problem just discussed. It is necessary to determine the load distribution $P(\xi)$ so that the integrals will be constant between a and -a. Also the functions involved are logarithmic. Then assuming that

$P(\xi)$ is distributed in the same manner as σ we have

$$P(\xi) = \frac{K}{2\pi\sqrt{a^2-x^2}}$$

where

$$K = \int_{-a}^{a} P(x)\,dx$$

is the total load on the boundary.

It is now necessary to check whether or not this does satisfy the boundary condition equation (31). For $y = o$ equation (38) becomes

$$v = \int_{-a}^{a} \frac{\varepsilon K s_{22}}{4\pi^2(\beta-\alpha)\sqrt{a^2-x^2}} \left(\frac{1}{\beta^2\varepsilon^2} - \frac{1}{\alpha^2\varepsilon^2}\right) \log(x-\xi)^2 d\xi$$

or remembering that $\alpha\beta = 1$

$$v = \frac{-K s_{22}(\alpha+\beta)}{4\pi^2\varepsilon} \int_{-a}^{a} \frac{\log(x-\xi)^2}{\sqrt{a^2-x^2}}\, d\xi .$$

In order to integrate this equation let $\xi = a\cos\phi$, then $d\xi = -a\sin\phi\, d\phi$ and

$$v = \frac{-K s_{22}(\alpha+\beta)}{4\pi^2\varepsilon} \int_{0}^{\pi} \log(x - a\cos\phi)^2 d\phi$$

or

$$v = \frac{-K s_{22}(\alpha+\beta)}{4\pi^2\varepsilon} \left[\int_{0}^{\pi} \log\left(1-\frac{a}{x}\cos\phi\right)^2 d\phi + \int_{0}^{\pi} \log x^2 d\phi\right]. \qquad (39)$$

The second integral is integrated immediately, but in order to find the value of the first integral consider the integral

$$\int_{-\pi}^{\pi} \log(1-k\cos\phi)^2 d\phi$$

where

$$k \geqq 1.$$

Then

$$\frac{d}{dk}\int_{-\pi}^{\pi} \log(1-k\cos\phi)^2 d\phi = \int_{-\pi}^{\pi} \frac{-2\cos\phi}{1-k\cos\phi}\, d\phi \quad .$$

This integral suggests the integration of the complex function

$$-\int_C \frac{z+\frac{1}{z}}{1-\frac{k}{2}\left(z+\frac{1}{z}\right)} \frac{dz}{iz}$$

around the unit circle in the z-plane since on the unit circle

$$z = e^{i\theta}, \; dz = e^{i\theta} i\, d\theta$$

and

$$z + \frac{1}{z} = 2\cos\theta$$

Rewriting the integral we have

$$-2i\int_C \frac{(1+z^2)}{z(kz^2-2z+k)}\, dz$$

which has singularities at

$$z = 0 \quad and \quad z = \frac{1}{k} \pm \frac{i}{k}\sqrt{k^2-1}\,.$$

Now

$$\left| \frac{1}{k} \pm \frac{i}{k}\sqrt{k^2-1} \right| = \frac{1}{k^2} + 1 - \frac{1}{k^2} = 1$$

which means that the last two singularities are on the unit circle. Each of these will be excluded from the contour with a circular arc of radius ε and with the center at the singularity. Now noticing that the angle θ passed through during the integration around these singularities is the same for each one since the radius ε is the same for each and letting

$$\frac{1}{k} + \frac{i}{k}\sqrt{k^2-1} = r_1$$

$$\frac{1}{k} - \frac{i}{k}\sqrt{k^2-1} = r_2$$

we have

$$\lim_{z \to r_1} (z - r_1) \frac{-2i(1+z^2)}{z(z-r_1)(z-r_2)} = \frac{-2i(1+r_1^2)}{r_1(r_1-r_2)}$$

and

$$\lim_{z \to r_2} (z - r_2) \frac{-2i(1+z^2)}{z(z-r_1)(z-r_2)} = \frac{+2i(1+r_2^2)}{r_2(r_1-r_2)}$$

Then the integral of f(z) around r_1 and r_2 is

$$-2\theta\left[\frac{1+r_1^2}{r_1(r_1-r_2)}\right] \quad and \quad 2\theta\left[\frac{1+r_2^2}{r_2(r_1-r_2)}\right]$$

respectively. The sum of these is

$$-2\theta\left[\frac{1+r_1^2}{r_1(r_1-r_2)} - \frac{1+r_2^2}{r_2(r_1-r_2)}\right] = -2\theta\left[\frac{r_2+r_2r_1^2-r_1-r_1r_2^2}{r_1r_2(r_1-r_2)}\right]$$

which reduces to

$$-2\theta\left[\frac{(r_1-r_2)(r_1r_2-1)}{r_1r_2(r_1-r_2)}\right] = 0$$

since

$$r_1r_2 = \frac{1}{k^2} + \frac{k^2-1}{k^2} = 1 .$$

Therefore using

$$\lim_{z\to 0} z\,\frac{-2i(1+z^2)}{z(kz^2-2z+k)} = \frac{-2i}{k}$$

we have

$$-2i\int_C \frac{1+z^2}{z(kz^2-2z+k)}\,dz = \frac{4\pi}{k}$$

or

$$\int_{-\pi}^{\pi} \frac{-2\cos\phi}{1-k\cos\phi}\, d\phi = \frac{4\pi}{k} .$$

The $\cos\phi$ is an even function so this reduces to

$$\int_{0}^{\pi} \frac{-2\cos\phi}{1-k\cos\phi}\, d\phi = \frac{2\pi}{k} .$$

Integrating this with respect to k in the following manner we have

$$\int_{1}^{k}\int_{0}^{\pi} \frac{-2\cos\phi}{1-k\cos\phi}\, d\phi\, dk = \int_{1}^{k} \frac{2\pi}{k}\, dk$$

or

$$\int_{0}^{\pi} \log(1-k\cos\phi)^2 d\phi \Big]_{1}^{k} = 2\pi \log k \Big]_{1}^{k} .$$

When k =1

$$\int_{0}^{\pi} \log(1-\cos\phi)^2 d\phi = 2\int_{0}^{\pi} \log\left(2\sin^2\frac{\phi}{2}\right) d\phi$$

which reduces to

$$2\int_{0}^{\pi} \left[\log 2 + \log^2\frac{\phi}{2}\right] d\phi .$$

Let $\phi = 2\lambda$, $d\phi = 2d\lambda$

then we have

$$2\pi \log 2 + 8\int_0^{\pi/2} \log \sin \lambda \, d\lambda$$

or

$$2\pi \log 2 - 4\pi \log 2 = -2\pi \log 2 .$$

Hence

$$\int_0^{\pi} \log (1 - k\cos\phi)^2 d\phi = 2\pi \log 2k$$

and equation (39) becomes

$$v = \frac{-K s_{22}(\alpha+\beta)}{4\pi^2 \varepsilon} \left[2\pi \log \frac{2a}{x} + 2\pi \log x \right]$$

or

$$v = \frac{-K s_{22}(\alpha+\beta)}{4\pi^2 \varepsilon} \log 2a .$$

Therefore the boundary condition equation (31) is satisfied, the load is the negative of the stress on the boundary and, since there is no load for $|x| > a$, equation (32) is satisfied, and the original stress function was determined so that there was no shear stress on the boundary, satisfying equation (30). The assumed load is therefore a solution to the problem.

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BIOGRAPHICAL NOTES

The author was born in Peoria, Illinois, on June 10, 1922. He received his elementary and secondary education in Florida public schools and graduated from Orlando Senior High School in 1940.

He enlisted in the Army of the United States in October, 1942. While in the service, he attended New York University for a period of four months. After his discharge from the Army in 1946, he enrolled in John B. Stetson University; and received the degree of Bachelor of Science, cum laude, in May, 1948.

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This dissertation was prepared under the direction of the chairman of the candidate's supervisory committee and has been approved by all members of the committee. It was submitted to the Dean of the College of Arts and Sciences and to the Graduate Council and was approved as partial fulfilment of the requirements for the degree of Doctor of Philosophy.

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